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Projections onto Hyperplanes in Banach Spaces

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1. INTRODUCTION AND NOTATION

In this paper X will always denote a real Banach space, X^* its norm dual, U, S (U^*, S^*) their unit balls and spheres. If V is a closed subspace of X, a projection onto V is a continuous linear operator $P: X \to V$ such that $Py = y$ if $y \in V$. A hyperplane in X is a subspace V of the form $V = f^{-1}(0)$, where $f \in S^*$. It is easy to see that any projection P onto the hyperplane $V = f^{-1}(0)$ is of the form $Px = x - f(x)z$, with $z \in f^{-1}(1)$; this projection will be denoted by P_z. We clearly have $1 \leqslant \|P_z\| \leqslant 1 + \|z\|$. Let $\varepsilon > 0$. Since $\exists\, z_\varepsilon \in f^{-1}(1)$ with $\|z_\varepsilon\| < 1 + \varepsilon$, we can always find a projection P with $\|P\| < 2 + \varepsilon$ and, when X is reflexive, with $\|P\| \leqslant 2$. The relative projection constant $\lambda(V, X)$ of V with respect to X is defined by: $\lambda(V, X) = \inf\{\|P\|: P$ projects X onto $V\}$; note that $1 \leqslant \lambda(V, X) \leqslant 2$; P is a minimal projection onto V if $\|P\| = \lambda(V, X)$. Reference [4] contains a very interesting and complete study of minimal projections and relative projection constants when X is one of the sequence spaces c_0, l^1.

The aim of this paper is to present some results related to the projections onto a hyperplane and to point out the relationships among the norms of the projections, the shape of the unit ball and the metric properties of the hyperplanes. Section 2 contains the main result (Theorem 3): it is proved that an upper bound for the number $\lambda(V, X)$ leads to the characterization of those hyperplanes which are range of a projection with norm strictly less than 2 (Theorem 4). In Section 3 an application of the previous results gives a substantial improvement of an inequality proved in [9] between the Jung constant J and the projection constants λ_1 of a Banach space (Theorem 6). In Section 4 a new parameter $F(X)$ of the Banach space X, depending on the collection of all hyperplanes of X, is considered and studied. Section 5 is devoted to a short investigation of the function ρ (defined below) and other functions related to the norm of the projections onto a given hyperplane.

We list now some other definitions and notations.

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For any real a set $V_a = f^{-1}(a)$ (note that all the V_a are isometric with $V = V_0$). For $0 \leqslant a < 1$ set $C_a = U \cap V_a$, $\Delta(a) = \frac{1}{2}$ diam C_a and

$$\rho(a) = \rho_V(a) = r_{V_a}(C_a) = r_V(C_a);$$

C_a is sometimes called a hypercircle, $\Delta(a)$ is half the diameter of the set C_a and $\rho(a)$ is the (Chebyshev) radius of C_a relative to the set V_a, i.e., the number:

$$\rho(a) = \inf_{z \in V_a} \sup\{\|z - x\|, \|x\| \leqslant 1, f(x) = a\}.$$

For $0 \leqslant \varepsilon$ set

$$E^\varepsilon(a) = \{x \in V_a \colon \sup_{y \in C_a} \|x - y\| \leqslant \rho(a) + \varepsilon\};$$

for $\varepsilon = 0$, $E^0(a) = E(a)$ is the (possibly empty) set of the centers of C_a relative to V_a. (Note that if $\varepsilon > 0$, $E^\varepsilon(a)$ is always non-empty.) C_a, $\rho(a)$ and $E^\varepsilon(a)$ are studied, in a slightly different situation, in [7].

2. Main Result

Let us begin with the following:

LEMMA 1. *Assume that* $0 \leqslant a < 1$, $\varepsilon \geqslant 0$.

(i) *For a* $c \in V_a$ *we have* $c \in E^\varepsilon(a)$ *if and only if*

$$\|c - y\| \leqslant \|y\| + \rho(a) - 1 + \varepsilon \tag{2.1}$$

for any $y \in C_a$.

(ii) $$1 - a \leqslant \rho(a) \leqslant 1 + a. \tag{2.2}$$

(iii) $$\|c\| \leqslant 2a + \rho(a) - 1 + \varepsilon \tag{2.3}$$

for any $c \in E^\varepsilon(a)$.

(iv) $$\Delta(a) \leqslant \rho(a) \leqslant (1 + a)\,\Delta(a). \tag{2.4}$$

Proof. (i) and (ii) are essentially Theorems 2 and 3 in [8]; for the sake of completeness we give here a new proof.

(i) Let $c \in V_a$. If (2.1) holds then clearly $\sup\{\|c - y\|, y \in C_a\} \leqslant \rho(a) + \varepsilon$ which means that $c \in E^\varepsilon(a)$. Assume now that $c \in E^\varepsilon(a)$ and that $x \neq c$ is a point in the relative interior of C_a; the line $\lambda c + (1 - \lambda) x$ meets the relative boundary of C_a in two points $\xi_i = \lambda_i c + (1 - \lambda_i) x$, with $\|\xi_i\| = 1$.

One of the λ_i, say λ_1, is strictly negative. We have $c-\xi_1=(1-\lambda_1)(c-x)$, so $(1-\lambda_1)\|c-x\| \leqslant \rho(a)+\varepsilon=\rho(a)-1+\|\xi_1\|+\varepsilon$. Since $\|\xi_1\| \leqslant \|x\|-\lambda_1\|x-c\|$ we get $(1-\lambda_1)\|c-x\| \leqslant \rho(a)-1+\|x\|-\lambda_1\|x-c\|$ so $\|x-c\| \leqslant \|x\|+\rho(a)-1$ and this last inequality holds also for points x in the relative boundary of C_a.

(ii) For $v \in V_a$ we have $\rho(a) \leqslant \sup\{\|v-y\|, y \in C_a\} \leqslant \|v\|+1$ which implies $\rho(a) \leqslant 1+a$ since $\inf\{\|v\|, v \in V_a\}=a$. For $c \in E^{\varepsilon}(a)$, $y \in C_a$ by (2.1) we have $\rho(a) \geqslant 1-\varepsilon+\|c-y\|-\|y\| \geqslant 1-\varepsilon-\|y\|$. This implies $\rho(a) \geqslant 1-a$. (Select z such that $\|z\|=1, f(z) \geqslant 1-\varepsilon$ and take $y=az/f(z)$.)

(iii) (2.3) is just a consequence of (2.1).

(iv) $\Delta(a) \leqslant \rho(a)$ is trivial. Let $x \in C_a$ with $\|x\|=1$, $v_\varepsilon \in V_a$ with $\|v_\varepsilon\|<a+\varepsilon$; there exists a $\lambda<0$ such that $\|\lambda x+(1-\lambda) v_\varepsilon\|=1$. So we have $1 \leqslant -\lambda+(1-\lambda)(a+\varepsilon)$; hence $1-\lambda \geqslant 2/(1+a+\varepsilon)$. $2\Delta(a) \geqslant \|x-(\lambda x+(1-\lambda) v_\varepsilon)\|=(1-\lambda)\|x-v_\varepsilon\|$. Taking sup on x we get $2\Delta(a) \geqslant (1-\lambda)\rho(a) \geqslant 2\rho(a)/(1+a+\varepsilon)$ which completes the proof of (2.4). ∎

Let us define

$$c=c_V=\sup\{\rho_V(a), 0 \leqslant a<1\}. \tag{2.5}$$

By (2.1) we have $1 \leqslant c_V \leqslant 2$. Define also $\gamma_z\colon [0,1) \to [0,\|P_z\|]$ by

$$\gamma_z(a)=\sup\{\|x-az\|, x \in C_a\}=\sup\{\|P_z x\|, x \in C_a\}, \tag{2.6}$$

where P_z is the usual projection defined by $P_z x=x-f(x) z$ $(f(z)=1)$. Clearly we have $\gamma_z(0)=1$, $\sup\{\gamma_z(a), 0 \leqslant a<1\}=\|P_z\|$; also, $\rho(a)=\inf_{v \in V_a} \sup\{\|x-v\|, x \in C_a\}=\inf_{z \in V_1} \sup\{\|x-az\|, x \in C_a\}=\inf_{z \in V_1}\|P_z\|=\inf_{z \in V_1} \sup_a \gamma_z(a) \geqslant \inf_z \sup_a \rho(a)=c_V=\sup_a \inf_z \gamma_z(a)$. We cannot in general interchange here inf sup with sup inf; i.e., in the inequalities

$$1 \leqslant c_V \leqslant \lambda(V, X) \tag{2.7}$$

it can happen that $c_V<\lambda(V, X)$. An example is given in Section 5. The parameter c_V is considered also in [9] but is defined differently; in [13] a related parameter $v(V)$ is studied. In order to make a comparison possible we note that, using the notations of [8, 9, 13] and the ones introduced here, we have the equivalences $\rho(a)=r(d/a)/(d/a)$, $r_s/s=\rho(d/s)$, where d is a fixed distance; see [8, 9, 13]. (This follows from the equality $C_1(s)=sC_{1/s}(1)$, where $C_a(s)=sU \cap V_a$.) In particular, note that

$$c_V=m(V, X) \qquad [9, \text{Lemma, p. } 42],$$

$$\rho'_+(0)=v(V)=\bar{1}((V)/d \qquad [13, \text{p. } 85];$$

here $\rho'_+(0) = \lim_{a\to 0^+}((\rho(a)-\rho(0))/a) = \lim_{a\to 0^+}((\rho(a)-1)/a)$. (The right derivative at the origin of ρ exists since the ratio $(\rho(a)-1)/a$ is non-increasing; see Section 5.) It is consequently easy to prove (see [13]) that

$$\lambda(V, X) \leqslant 1 + \rho'_+(0) = 1 + \nu(V). \tag{2.8}$$

We now want to prove a lemma on projections.

Let $V = f^{-1}(0), f \in S^*$, $0 < a < 1$ and $\varepsilon > 0$; select $z_\varepsilon \in f^{-1}(1)$ such that $\|z_\varepsilon\| < 1 + \varepsilon$ and $c_a^{a\varepsilon} \in E^{a\varepsilon}(a)$. We define the projections P_{z_ε} and Q_a^ε onto V by

$$P_{z_\varepsilon} x = x - f(x)\, z_\varepsilon, \qquad Q_a^\varepsilon x = x - f(x)\, c_a^{a\varepsilon}/a.$$

Set also $A = \{x \in S\colon a \leqslant f(x) \leqslant 1\}$, $B = \{x \in S\colon 0 \leqslant f(x) < a\}$.

LEMMA 2. *We have*

$$\sup_{x\in A} \|P_{z_\varepsilon} x\| < 2 + \varepsilon, \qquad \sup_{x\in B} \|P_{z_\varepsilon} x\| < 1 + a + \varepsilon,$$

$$\|P_{z_\varepsilon}\| < 2 + \varepsilon, \tag{2.9}$$

$$\sup_{x\in A} \|Q_a^\varepsilon x\| \leqslant 1 + (\rho(a)-1)/a + \varepsilon, \qquad \sup_{x\in B} \|Q_a^\varepsilon x\| \leqslant 2a + \rho(a) + \varepsilon,$$

$$\|Q_a^\varepsilon\| \leqslant \max(1 + (\rho(a)-1)/a,\ 2a + \rho(a)) + \varepsilon. \tag{2.10}$$

Proof. (2.9) is trivial. Let us prove (2.10). If $x \in A$ then $ax/f(x) \in C_a$; therefore

$$\|Q_a^\varepsilon x\| = \left\| \frac{f(x)}{a} \left[\frac{a}{f(x)} x - c_a^{a\varepsilon} \right] \right\|.$$

Using (2.1) we obtain

$$\begin{aligned} \|Q_a^\varepsilon x\| &\leqslant \frac{f(x)}{a} \left[\frac{a}{f(x)} \|x\| + \rho(a) - 1 + a\varepsilon \right] \\ &\leqslant 1 + \frac{f(x)}{a} [\rho(a) - 1 + a\varepsilon]. \end{aligned}$$

If $x \in B$, using (2.3) we obtain $\|Q_a^\varepsilon x\| \leqslant 1 + \|c_a^{a\varepsilon}\| \leqslant 2a + \rho(a) + \varepsilon$. ∎

Let us consider for $0 < a < 1$ the function ψ defined by

$$\psi(a) = \left(1 + \frac{\rho(a)-1}{a}\right) - (2a + \rho(a)) = \frac{(\rho(a)-1)(1-a)}{a} - 2a. \tag{2.11}$$

Since $\rho(a) \leqslant 1 + a$ we have $\psi(a) \leqslant 1 - 3a$; hence $\psi(a) < 0$ if $a > \frac{1}{3}$. Also $\psi(0) = \rho'_+(0)$. By (2.8) we have $\psi(0) > 0$ if $\lambda(V, X) > 1$; therefore in this case there exists a $\beta \in (0, \frac{1}{3}]$ such that $\psi(\beta) = 0$. Recalling that $c = \sup \rho(a)$ we also have $\psi(a) \leqslant (c-1)(1-a)/a - 2a$. Assume that $1 < c < 2$; then $\psi(2(c-1)/c) \leqslant (8 - 6c - c^2)/2c$ and therefore $\psi(2(c-1)/c) < 0$ if $c > \sqrt{17} - 3 = 1.123\ldots$. We shall use this last fact in proving Theorem 2.

Now consider the problem: when does a projection $P: X \to V$ exist with $\|P\| \leqslant 2$? If $\lambda(V, X) < 2$ this is obviously the case; when $\lambda(V, X) = 2$ this is still the case if X is reflexive. We shall prove a more general result.

Recall that it is said that a Banach space X admits centers if for every bounded subset A of X the set of the (absolute) centers of A is non-empty. Examples of such spaces are: dual (hence reflexive) spaces, $L^1(\mu)$ (μ σ-finite) and $C(Q)$ (Q Hausdorff compact) but the class is wider; see [2] for new examples and a survey of the classical existence theorems.

Let us consider V as a Banach space in itself; noting that $E(a)$ is the set of the centers of C_a in V_a which is isometric with V, we see that $E(a)$ is non-empty if V admits centers.

THEOREM 1. *If V admits centers there exists a projection $P: X \to V$ such that $\|P\| \leqslant 2$.*

Proof. Since V admits centers $E(a)$ is non-empty for $0 \leqslant a < 1$; the projection Q_a^ε considered in Lemma 2 is defined also for $\varepsilon = 0$ by any $c_a \in E(a)$ $(Q_a x = x - c_a f(x)/a)$. Using (2.10) we get $\|Q_a\| \leqslant \max(1 + (\rho(a) - 1)/a,\ 2a + \rho(a))$. Since we may assume that $\lambda(V, X) > 1$ there exists a $\beta \in (0, \frac{1}{3})$ such that $\psi(\beta) > 0$. For this β we have $\|Q_\beta\| \leqslant 1 + (\rho(\beta) - 1)/\beta$; hence $\|Q_\beta\| \leqslant 2$ since $\rho(\beta) \leqslant 1 + \beta$. ∎

EXAMPLE 1. Take $X = 1^1$, $V = f^{-1}(0)$, where $f \in S^*$ is the element of 1^∞ defined by $f = (1/2, 2/3, \ldots, (n-1)/n, \ldots)$. We have $\lambda(V, X) = 2$ (see [4, Corollary, p. 224]). On the other hand, it is easy to see that there is no norm 2 projection onto V; therefore for any $P: X \to V$ we have $\|P\| > 2$ (no projection is minimal). This counterexample is due to Grünbaum [15, p. 199]. By the preceding theorem V does not admit centers. For a similar negative example see [10, p. 41].

We now prove our main result.

THEOREM 2. *Let $V = f^{-1}(0)$, $f \in S^*$. For every $\sigma > 0$ there exists a projection $P_\sigma: X \to V$ such that*

$$\|P_\sigma\| \leqslant g(c) + \sigma, \tag{2.12}$$

where c is defined by (2.5) *and* g: $[1, 2] \to [1, 2]$ *by*

$$g(c) = 1 + \tfrac{1}{2}\{(c-1) + \sqrt{(c-1)^2 + 8(c-1)}\} \qquad \text{if} \quad 1 \leqslant c \leqslant \sqrt{17} - 3$$

$$= 1 + \frac{8(c-1)}{c^2 + 4(c-1)} \qquad \text{if} \quad \sqrt{17} - 3 < c \leqslant 2. \tag{2.13}$$

Proof. By (2.10) we have for the projection Q_a^σ: $\|Q_a^\sigma\| \leqslant \max(1 + (\rho(a) - 1)/a,\ 2a + \rho(a)) + \sigma \leqslant \max(1 + (c-1)/a,\ 2a + c) + \sigma$.

Computing the optimal value for a we find a projection Q^σ such that $\|Q^\sigma\| \leqslant 1 + \frac{1}{2}\{(c-1) + \sqrt{(c-1)^2 + 8(c-1)}\} + \sigma$. (This computation was done in [9, Theorem 4]. We will give here a much better result when $c > \sqrt{17} - 3$.)

Set $P_\lambda^\sigma = \lambda P_{z_\sigma} + (1-\lambda) Q_a^\sigma$. P_λ^σ is of course a projection and for $0 \leqslant \lambda \leqslant 1$, using Lemma 2, we have

$$\begin{aligned} \|P_\lambda^\sigma x\| &\leqslant \lambda \|P_{z_\sigma} x\| + (1-\lambda)\|Q_a^\sigma x\| \\ &\leqslant \lambda(2+\sigma) + (1-\lambda)\left(1 + \frac{\rho(a)-1}{a} + \sigma\right) \qquad \text{if} \quad x \in A \\ &\leqslant \lambda(1 + a + \sigma) + (1-\lambda)(2a + \rho(a) + \sigma) \qquad \text{if} \quad x \in B \end{aligned}$$

hence we obtain

$$\|P_\lambda^\sigma\| \leqslant \max\left(2\lambda + (1-\lambda)\frac{a + \rho(a) - 1}{a},\ (1+a)\lambda + (1-\lambda)(2a + \rho(a)) + \sigma\right).$$

When $\psi(a) < 0$ (ψ is defined by (2.11)) a possible and optimal choice for λ in $[0, 1]$ is

$$\lambda = \lambda_a = -\frac{\psi(a)}{(1-\rho(a))/a + a + \rho(a)} = \frac{2a^2 + (1-a)(1-\rho(a))}{1 + a^2 - \rho(a)(1-a)}.$$

With such a choice we get

$$\|P_{\lambda_a}^\sigma\| \leqslant 1 + \frac{2a^2}{1 + a^2 - (1-a)\,\rho(a)} + \sigma \leqslant 1 + \frac{2a^2}{1 + a^2 - (1-a)\,c} + \sigma.$$

We have seen that $\psi((2c-2)/c) < 0$ if $c > \sqrt{17} - 3$; therefore the choice $a = (2c-2)/c$ is permitted if $\sqrt{17} - 3 < c < 2$ (it must be $a < 1$) and we obtain a projection R^σ such that $\|R^\sigma\| \leqslant 1 + 8(c-1)/(c^2 + 4(c-1)) + \sigma$. Using Q^σ and R^σ the proof of this theorem is completed (note that when $c = 2$, $g(c) = 2$ and (2.12) holds). ∎

Remark. The function g has the following properties: $g \in C^1(1,2)$; $g(1)=1$, $g(2)=2$; $c \leqslant g(c)$; g is strictly increasing and concave; $g'(1)=+\infty$, $g'(2)=0$. In the point $c_0=\sqrt{17}-3$ we have $g(c_0)=(\sqrt{17}-1)/2$, $g'(c_0)=(\sqrt{17}+1)/2$.

THEOREM 3. *We have*

$$1 \leqslant c_V \leqslant \lambda(V,X) \leqslant g(c_V) \leqslant 2, \tag{2.14}$$

where the function g is defined by (2.13).

Proof. This is (2.7) and an obvious consequences of Theorem 2. ▮

THEOREM 4. *We have*

$$\lambda(V,X)=1 \Leftrightarrow c_V=1 \Leftrightarrow \forall\, a \in (0,1)\colon \rho(a) \leqslant 1, \tag{2.15}$$

$$\lambda(V,X)<2 \Leftrightarrow c_V<2 \Leftrightarrow \exists\, a \in (0,1)\colon \rho(a) < 1+a. \tag{2.16}$$

Proof. (2.15) follows from (2.14) since $g(1)=1$. By (2.14) and the properties of g it follows that $c_V<2 \Leftrightarrow \lambda(V,X)<2$. Also, $c_V<2 \Rightarrow \exists\, a\colon \rho(a)<1+a$. Assume now that for a $\beta \in (0,1)$ we have $\rho(\beta)<1+\beta$. If $\psi(\beta) \geqslant 0$ for σ small enough the projection Q_a^σ used in Theorem 2 has norm $\|Q_\beta^\sigma\|<2$ for $a=\beta$. If $\psi(\beta)<0$ the projection $P_{\lambda_\beta}^\sigma$ (see Theorem 2) has norm $\|P_{\lambda_\beta}^\sigma\| \leqslant 1+\beta^2+\sigma$. ▮

3. THE PARAMETERS J AND λ_1

We first recall briefly some well known definitions and properties of certain projection constants. Assume that V is a real Banach space: we say that $V \in \mathscr{P}_\lambda$ $(\lambda \geqslant 1)$ if for every superspace Z there is a projection $P\colon Z \to V$ such that $\|P\| \leqslant \lambda$. The (absolute) projection constant of V is $\lambda(V)=\inf\{r\colon V \in \mathscr{P}_r\}$. We say that $V \in E_\lambda$ $(\lambda \geqslant 1)$ if for every superspace Z with $\dim Z/V=1$ there is a projection $P\colon Z \to V$ such that $\|P\| \leqslant \lambda$. The constant $\lambda_1(V)$ is defined by $\lambda_1(V)=\inf\{r\colon V \in E_r\}$.

Note that $\lambda(V)=\sup\{\lambda(V,Z)\colon V \subset Z\}$ and

$$\lambda_1(V)=\sup\{\lambda(V,Z)\colon V \subset Z,\ \dim Z/V=1\}. \tag{3.1}$$

It is easily seen that $1 \leqslant \lambda_1(V) \leqslant \lambda(V) \leqslant \infty$, $\lambda_1(V) \leqslant 2$. We recall also the definition of the Jung constant of V, $J(V)$:

$$J(V)=\sup\{r(A)/\Delta(A),\ A \subset V,\ A \text{ bounded}\};$$

here $r(A)$ is the (absolute Chebyshev) radius of A and $\Delta(A)=\frac{1}{2}\operatorname{diam}(A)$.

Clearly $1 \leqslant J(V) \leqslant 2$. References on all these parameters are found in [9] where especially the relationship between J and λ_1 is investigated.

We now give some applications of the results of Section 2.

We note that when V is a hyperplane in X, for greater precision one should write $c(V, X)$ instead of c_V or c and $\rho_{V,X}(a)$ instead of $\rho_V(a)$ or $\rho(a)$.

Theorem 3 in [9] can be stated as:

THEOREM 5 (see [9]).

$$J(V) = \sup\{c(V, X),\ V \subset X,\ \dim X/V = 1\}. \tag{3.2}$$

The following is the main application, in this context, of Theorem 3.

THEOREM 6. *We have*

$$1 \leqslant J(V) \leqslant \lambda_1(V) \leqslant g(J(V)) \leqslant 2. \tag{3.3}$$

Proof. In (2.14) take $\dim X/V = 1$, use (3.1), (3.2) and the fact that g is strictly increasing. ∎

COROLLARY (see [7]).

$$J(V) = 1 \Leftrightarrow \lambda_1(V) = 1.$$

This was first proved in [7]; see [9] for other equivalences and references.

Theorem 7 follows immediately from Theorem 6.

THEOREM 7. $J(V) = 2 \Leftrightarrow \lambda_1(V) = 2$.

This is a new result. This theorem has motivations in Banach space theory; see, for example, Theorem 8. The interest in describing situations where the Jung constant and the projection constant λ_1 have the same value goes back to Grünbaum (see [14, 15]).

We remark that Theorem 6 is a substantial improvement of Theorem 4 (formula (4)) in [9] since the new bound $\lambda_1(V) \leqslant g(J(V))$ is now significant for every value of $J(V)$. This fact gives a parallel improvement of Theorem 5 in [9]. In fact we have:

THEOREM 8. *Let $C(Q)$ be the space of real continuous functions on the compact Q with the usual* sup norm. *We have*

$$J(C(Q)) < 2 \Leftrightarrow C(Q) \in \mathscr{P}_1.$$

Proof. If $J(C(Q)) < 2$ by (3.3), $\lambda_1(C(Q)) < 2$ and this implies that $C(Q) \in \mathscr{P}_1$ by a theorem of Amir; see [1]. ∎

This last result has been proved independently by Professor Amir who communicated it at the 1981 meeting on Approximation Theory in Oberwolfach.

Note that $C(Q) \in \mathscr{P}_1$ if and only if Q is stonian.

4. The Parameter F

We now discuss the relevance of the previous results from a different point of view. For a given (real) Banach space X let us define

$$F(X) = \sup\{\lambda(V, X),\ V \text{ is a hyperplane in } X\}.$$

If $\dim X = 1$, $F(X) = 0$ and if $\dim X = 2$, $F(X) = 1$. To avoid trivialities we assume in this section that $\dim X > 2$.

F is a parameter of the space which satisfies $1 \leqslant F(X) \leqslant 2$. If X is a Hilbert space of course $F(X) = 1$. For the converse observe that the classical Kakutani's theorem (X is Hilbert if and only if every hyperplane V in X is range of a norm one projection) is not applicable here since the condition $\lambda(V, X) = 1$ does not imply, in general, the existence of a norm one projection onto V; however, still the condition $F(X) = 1$ implies that X is a Hilbert space. This fact was pointed out to me by Professor Amir and can be proved using the Garkavi–Klee characterization of Hilbert spaces via Chebyshev centers.

How to evaluate $F(X)$? Again Theorem 3 turns out to be useful. Define

$$C(X) = \sup\{c_V,\ V \text{ is a hyperplane of } X\}.$$

We easily obtain the analog of Theorems 3 and 4, namely,

THEOREM 9. *For the Banach space X we have*:

$$1 \leqslant C(X) \leqslant F(X) \leqslant g(C(X)) \leqslant 2, \tag{4.1}$$

$$F(X) = 1 \Leftrightarrow C(X) = 1, \tag{4.2}$$

$$F(X) < 2 \Leftrightarrow C(X) < 2. \tag{4.3}$$

We give now, in a particular case, a more precise evaluation. Recall that in a Banach space X the modulus of convexity of X is the function $\delta_X: [0, 2] \to [0, 1]$ defined by $\delta_X(\varepsilon) = \inf\{1 - \|x + y\|/2\colon\ x, y \in S, \|x - y\| \geqslant \varepsilon\}$. X is uniformly convex (u.c.) if and only if $\delta_X(\varepsilon) > 0$ for $\varepsilon > 0$; in this case δ_X is invertible and we denote by η_X the inverse function. Assume that X is u.c., $V=f^{-1}(0)$, $f\in S^*$, $0\leqslant a\leqslant 1$ and set $\Gamma_a=\{x\in S: f(x)\geqslant a\}\supset C_a$. We have the following simple result:

$$\Delta(a)\leqslant \operatorname{diam}\Gamma_a/2\leqslant \eta_X(1-a)/2. \tag{4.4}$$

In fact, assume that $x, y\in\Gamma_a\cap S$; then $\|x+y\|/2\leqslant 1-\delta_X(\|x-y\|)$, that is, $\delta_X(\|x-y\|)\leqslant 1-\|x+y\|/2\leqslant 1-a$; hence $\delta_X(\operatorname{diam}\Gamma_a)\leqslant 1-a$ which implies (4.4).

THEOREM 10. *If V is a hyperplane in a u.c. space X we have*

$$\rho_V(a)\leqslant \eta_X(1-a)(1+a)/2. \tag{4.5}$$

Consequently

$$\begin{aligned} C(X)&\leqslant \sup_a \eta_X(1-a)(1+a)/2=D(X)<2,\\ F(X)&\leqslant g(D(X))<2. \end{aligned} \tag{4.6}$$

Proof. (4.5) follows from (2.4) and (4.4), then observe that the right hand side of (4.5) does not depend on V; hence (4.6) follows immediately using Theorem 3. ∎

Note that from (4.4) it follows the well known fact that in a u.c. space we have $\lim_{a\to 1^-}\rho(a)=\lim_{a\to 1^-}\Delta(a)=0$.

The fact that $F(X)<2$ in a u.c. space X is contained in a more general result that we will prove in Theorem 12. We need first to recall some other facts on Banach spaces.

A Banach space X is uniformly non-square (u.n.s.) if there exists an $\varepsilon>0$ such that $\min(\|x+y\|, \|x-y\|)\leqslant 2-\varepsilon$ for $x, y\in U$. It is easily seen that if X is u.c. then X is u.n.s.

The radial projection $R: X\to U$ is defined by

$$\begin{aligned} Rx&=x && \text{if } x\in U\\ &=x/\|x\| && \text{if } x\notin U. \end{aligned}$$

The radial constant $k(X)$ of the real Banach space X is defined by

$$k(X)=\sup\left\{\frac{\|Rx-Ry\|}{\|x-y\|},\qquad x, y\in X,\ x\neq y\right\}.$$

It is well known that $1\leqslant k(X)\leqslant 2$; see, for example, [11] where other properties of k are also described. Thiele proved in [18] the interesting fact that $k(X)<2\Leftrightarrow X$ is u.n.s.

Smith introduced in [16] the metric projection bound $MPB(X)$ of the space X by $MPB(X) = \sup\{\|P_M\|, M$ is a proximinal subspace of $X\}$, where $P_M(x) \subset M$ is the set of best approximations of x in M (non-empty by definition when M is proximinal) and $\|P_M\| = \sup\{\|y\|, y \in P_M(x), \|x\| \leqslant 1\}$.
Baronti proved in [3] that $MPB(X) = k(X)$.
Collecting all these facts we are able to prove:

THEOREM 11. *For any real Banach space X we have*

$$F(X) \leqslant k(X). \tag{4.7}$$

Proof. Set $\overline{MPB}(X) = \sup\{\|P_V\|: \dim X/V = 1, V \text{ proximinal}\}$. Obviously $\overline{MPB}(X) \leqslant MPB(X)$. (4.7) will be proved showing that $F(X) \leqslant \overline{MPB}(X)$. First note that u.n.s. Banach spaces are reflexive (this is a well known result due to R. C. James) so that the condition $k(X) < 2$ implies reflexivity: (4.7) is therefore trivially true if X is not reflexive since $k(X) = 2$. Assume that X is reflexive and consequently that any hyperplane V is proximinal in X: the (multivalued) best approximation operator P_V admits always a continuous linear selection which is therefore a projection. The inequality $F(X) \leqslant \overline{MPB}(X)$ will follow from the definitions of $\overline{MPB}(X)$ and of $\lambda(V, X)$. ∎

We recall now a useful result of Bohnenblust (see [5]): let V be a hyperplane in an n-dimensional space X. There always exists a projection $P: X \to V$ such that $\|P\| \leqslant 2(n-1)/n$. This means that

$$\dim X = n \Rightarrow F(X) \leqslant 2 - 2/\dim X. \tag{4.8}$$

Combining (4.7), (4.8) and Thiele's theorem already mentioned we get:

THEOREM 12. *We have $F(X) < 2$ in the following cases: X is finite dimensional, X is uniformly non-square.*

5. THE FUNCTIONS ρ, γ_z AND Δ

Let the hyperplane $V = f^{-1}(0)$ be fixed in X, $z \in f^{-1}(1)$, $P_z: X \to V$ defined by $P_z x = x - f(x)\, z$ and $0 \leqslant a < 1$. This section is devoted to a short study of the following functions:

$$\rho(a) = \inf_{v \in V_a} \sup\{\|x - v\|: x \in C_a\} = r_{V_a}(C_a) = r_V(C_a),$$

$$\gamma_z(a) = \sup\{\|x - az\|: x \in C_a\},$$

$$\Delta(a) = \tfrac{1}{2} \sup\{\|x - y\|: x, y \in C_a\} = \tfrac{1}{2} \operatorname{diam}(C_a).$$

Recall that

$$\Delta(a) \leqslant \rho(a) = \inf_{z \in V_1} \gamma_z(a); \qquad \|P_z\| = \sup_a \gamma_z(a).$$

Denote now by φ any of the functions ρ, γ_z, Δ. We shall prove below that $a \to \varphi(a)/a$ is non-increasing in $(0, 1)$; therefore we can define $\varphi(1) = \lim_{a \to 1^-} \varphi(a)$.

From now on we will consider φ as defined in the closed interval $[0, 1]$. Note that $\varphi(0) = 1$ and that when X is u.c. $\varphi(1) = 0$ (see (4.4)). Let us prove:

THEOREM 13. *For $\alpha \leqslant \beta$, $\alpha \neq 1$, we have*

$$-\frac{\varphi(\alpha)}{1-\alpha}(\beta - \alpha) \leqslant \varphi(\beta) - \varphi(\alpha) \leqslant \frac{\varphi(\alpha) - 1}{\alpha}(\beta - \alpha). \tag{5.1}$$

Moreover the function φ is continuous in $[0, 1]$ and Lipschitz in every interval $[0, 1 - \varepsilon]$ with $\varepsilon > 0$.

Proof. For $s > 0$ set $C_a^s = V_a \cap sU$. We generalize the functions φ by putting $\varphi^s(a) = \varphi(C_a^s)$ (to be defined in the natural way). Note that $\varphi^1 = \varphi$. It is easy to see that for $h > 0$ we have

$$\varphi^{s+h}(a) \geqslant \varphi^s(a) + h. \tag{5.2}$$

Let T be the map $x \to \alpha/\beta\, x$; then $TC_\beta^1 \subset C_\alpha^{\alpha/\beta}$ since $\|Tx - Ty\| = \alpha/\beta\, \|x - y\|$. Using (5.2) we get $\alpha/\beta\, \varphi^1(\beta) \leqslant \varphi^{\alpha/\beta}(\alpha) \leqslant \varphi^1(\alpha) - (1 - \alpha/\beta)$, that is, $\varphi(\alpha) \geqslant (1 - \alpha/\beta) + \alpha/\beta\, \varphi(\beta)$ which is the right hand side of (5.1). Let Z be the map $x \to \lambda x + (1 - \lambda) z$, $\lambda \in [0, 1]$, $f(z) = 1$. We have $ZC_\alpha^1 \subset C_{\lambda\alpha + (1-\lambda)}^{\lambda + (1-\lambda)\|z\|}$ and, taking the infimum on the z with $f(z) = 1$, also $ZC_\alpha^1 \subset C_{\lambda\alpha + (1-\lambda)}^{\lambda + (1-\lambda)} = C_{\lambda\alpha + (1-\lambda)}^1$. Since $\|Zx - Zy\| = \lambda\, \|x - y\|$ we obtain $\lambda\varphi(\alpha) \leqslant \varphi(\lambda\alpha + (1 - \lambda))$ which gives, for $\beta = \lambda\alpha + (1 - \lambda)$, $((1 - \beta)/(1 - \alpha))$ $\varphi(\alpha) \leqslant \varphi(\beta)$ which is the left hand side of (5.1).

The other conclusions of the theorem follow immediately from (5.1). ∎

Remarks. The right hand side of (5.1) may be written $\varphi(\alpha) \geqslant (1 - \alpha/\beta) + \alpha/\beta\, \varphi(\beta)$ which in particular means that the hypograph of φ is convex with respect to the point $(0, \varphi(0)) = (0, 1)$: we will say that φ is concave with respect to 0.

We also have $\varphi(\alpha)/\alpha - \varphi(\beta)/\beta \geqslant (\beta - \alpha)/\alpha\beta$, i.e., $\alpha \to \varphi(\alpha)/\alpha$ is non-increasing, and, more significantly, we also have

$$\frac{\varphi(\alpha) - 1}{\alpha} - \frac{\varphi(\beta) - 1}{\beta} \geqslant -\frac{1}{\alpha} + \frac{1}{\beta} + \frac{\beta - \alpha}{\alpha\beta} = 0,$$

i.e., $\alpha \to (\varphi(\alpha) - 1)/\alpha = (\varphi(\alpha) - \varphi(0))/\alpha$ is non-increasing, or equivalently

$$\frac{\varphi(\beta) - \varphi(\alpha)}{\beta - \alpha} \leqslant \frac{\varphi(\alpha) - 1}{\alpha} \qquad \text{for} \quad \alpha \leqslant \beta.$$

We set $\lim_{\alpha \to 0^+} ((\varphi(\alpha) - 1)/\alpha) = \varphi'_+(0)$. Note that $(\varphi(\beta) - 1)/\beta \leqslant \varphi'_+(0)$, i.e., $\varphi(\beta) \leqslant \varphi'_+(0)\,\beta + 1$ and $\varphi(\beta) - \varphi(\alpha) \leqslant (\beta - \alpha)\,\varphi'_+(0)$. We can also see from (5.1) that $\alpha \to \varphi(\alpha)/(1 - \alpha)$ is non-decreasing, that φ is concave with respect to 1 if $\varphi(1) = 0$ (for example, when X is u.c.) and finally that φ is non-increasing in the set $\{x \in [0, 1]: \varphi(x) \leqslant 1\}$ and $\varphi(x) \geqslant 1$ in $[0, \xi]$, $\varphi(x) < 1$ in $(\xi, 1]$, where $\xi = \sup\{x: \varphi(x) \geqslant 1\}$.

For $a = 1$ the set $C_1 = \{x \in S: f(x) = 1\}$ may of course be empty (for this reason the functions φ where defined originally only in $[0, 1)$). If we assume that $C_1 \neq \varnothing$ we can define $\varphi_1 = \varphi(C_1)$. It is easy to see that $\varphi_1 \leqslant \varphi(1)$. We give an example where the inequality is strict.

EXAMPLE 2. Let X be 1^1, $V_p = f_p^{-1}(0)$ with

$$f_p = (\underbrace{1, 1, \ldots, 1}_{p\text{ terms}}, 1/2, 2/3, \ldots, (n-1)/n, \ldots).$$

By [4, Corollary, p. 224], we have $\lambda(V_p, X) = 2$; consequently by (2.16) $\sup \rho(a) = 2$ and $\rho(1) = 2$. However, one can see that $\rho_1 = 0$ for $p = 1$ and $\rho_1^a \leqslant 1$ for $p = 2$; here $\rho_1 = r_{V_p}(C_1)$.

It could be asked whether the functions φ are concave. We will show with an example that this is not the case when $\varphi = \gamma_z$. It is, in general, difficult to compute explicitly the functions φ; however, when X is a space of continuous functions, this is sometimes possible using an interesting and useful formula due to Smith and Ward [17].

THEOREM 14 (see [12, 17]).
Let T be a topological space, Y a subset of $C(T)$, and A a bounded subset of $C(T)$. Then

$$r_Y(A) = r(A) + d(Y, E(A)). \tag{5.3}$$

Here $r(A)$, $r_Y(A)$ are, respectively, the absolute radius and the radius with respect to Y of the set A, $d(Y, E(A))$ is the distance from Y of the (non-empty) set of the absolute centers of A.

The formula (5.3) was proved by Smith and Ward for T paracompact; the extension to any topological T is given in [12], where also a different proof and several applications of this formula are given. For the classical formulas for $r(A)$ and $E(A)$ in $C(T)$ see, for example, [12]. Note that for $Y = V = f^{-1}(0)$, $A = C_a$, $0 \leqslant a \leqslant 1$, we have

$$\rho(a) = r(C_a) + d(V_a, E(C_a)). \tag{5.4}$$

EXAMPLE 3. Let $X = 1^\infty(3)$, $V = f^{-1}(0), f = (\frac{3}{8}, \frac{1}{4}, \frac{3}{8})$. One can see that

$$\begin{aligned} d(V_a, E(C_a)) &= a && \text{for} \quad 0 \leqslant a \leqslant \tfrac{1}{4} \\ &= \tfrac{1}{2} - a && \tfrac{1}{4} < a \leqslant \tfrac{1}{2} \\ &= 0 && \text{for} \quad \tfrac{1}{2} < a \leqslant 1, \\ r(C_a) &= 1 && \text{for} \quad 0 \leqslant a \leqslant \tfrac{1}{2} \\ &= 2 - 2a && \text{for} \quad \tfrac{1}{2} < a \leqslant 1. \end{aligned}$$

By (5.4) we obtain

$$\begin{aligned} \rho(a) &= 1 + a && \text{for} \quad 0 \leqslant a \leqslant \tfrac{1}{4} \\ &= \tfrac{3}{2} - a && \text{for} \tfrac{1}{4} < a \leqslant \tfrac{1}{2} \\ &= 2 - 2a && \text{for} \quad \tfrac{1}{2} < a \leqslant 1; \end{aligned}$$

hence $\sup \rho(a) = c_V = \rho(1/4) = 5/4$.

On the other hand, we have (see [4, Theorem 2], also [6, Theorem 3])- $\lambda(V, X) = 9/7$, so this is a case of strict inequality in (2.7). Note that in this example the function ρ is concave.

We consider now a minimal projection: let $z = (8/7, 4/7, 8/7)$ (note that $\|z\| = 8/7 > 1$). The projection P_z ($P_z x = x - f(x)z$) is minimal since $\|P_z\| = \sup_a \gamma_z(a) = 9/7$. In fact: $\gamma_z(a) = \sup\{\|x - az\|, f(x) = a, \|x\| \leqslant 1\}$. Letting $x = (x_1, x_2, x_3)$ we have $x - az = (x_1 - a8/7, x_2 - a4/7, x_3 - a8/7)$ with the conditions $|x_i| \leqslant 1$, $3x_1 + 2x_2 + 3x_3 = 8\text{a}$. For $0 \leqslant a \leqslant 1/4$ choosing $x = (-1, 4a, 1)$ we get $\gamma_z(a) = 1 + a8/7$; for $a = 1/2$ choosing $x = (1, -1, 1)$ we get $\gamma_z(1/2) = 9/7$. Also $\gamma_z(1) = 3/7$. Finally note that $x - az = 1/14(8x_1 - 4x_2 - 6x_3, \ -3x_1 + 12x_2 - 3x_3, \ -6x_1 - 4x_2 + 8x_3)$; therefore $\gamma_z(a) \leqslant 9/7$ and equality is possible only for x of the form $\pm(1, -1, -1)$, $\pm(-1, 1, -1)$, $\pm(-1, -1, 1)$. Since $8a \geqslant 0$ the choice reduces to $(-1, 1, 1)$, $(1, -1, 1)$, $(1, 1, -1)$ corresponding to the values 1/4, 1/2, 1/4 for a. We conclude that $\gamma_z(a) < 9/7$ if $a \notin \{1/4, 1/2\}$. We have shown that the function γ_z cannot be concave.

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